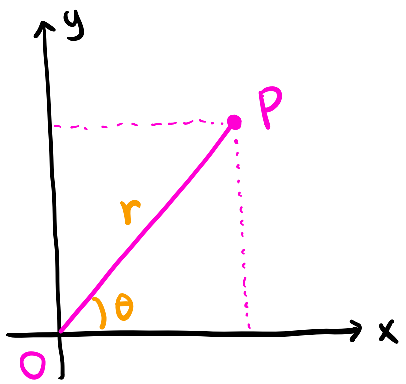
15.3. Double integrals in polar coordinates

Def Polar coordinates are related to rectangular coordinates by $x = r\cos\theta$ and $y = r\sin\theta$.



r : the distance from $O = (0,0)$

$\Rightarrow r = \sqrt{x^2+y^2}$.

θ : the angle between OP and the positive x-axis.

$\Rightarrow \tan\theta = \dfrac{y}{x}$

Note The angle θ is measured counterclockwise.

Prop If $f(x,y)$ is a continuous function on a domain D,

"Jacobian"

then $$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta,\, r\sin\theta)\, r\,dr\,d\theta$$

where the bounds on the right sides are given in polar coordinates.

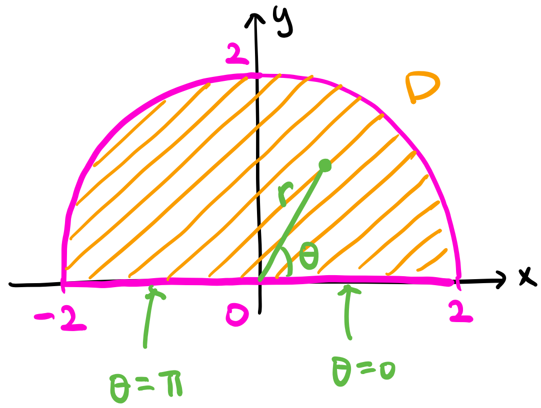
Recall: For single variable integrals, a substitution $u = g(x)$ introduces an extra factor $g'(x)$. ($du = g'(x)\,dx$)

Note For double integrals, a conversion to a different coordinate system is essentially a two-dimensional substitution, and thus introduces an extra factor called the Jacobian.

* This topic will be further discussed in Lab 4.

Ex Evaluate $\iint_D x^2 y\, dA$ where D is the top half of the disk with center $(0,0)$ and radius 2.

Sol



In polar coordinates, D is given by

$0 \le \theta \le \pi$ and $0 \le r \le 2$.

Jacobian

$$\iint_D x^2 y\, dA = \int_0^{\pi}\int_0^{2} (r\cos\theta)^2 (r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{2} r^4 \cos^2\theta \sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{r^5}{5}\cos^2\theta\sin\theta \Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi} \frac{32}{5}\cos^2\theta \sin\theta\, d\theta$$

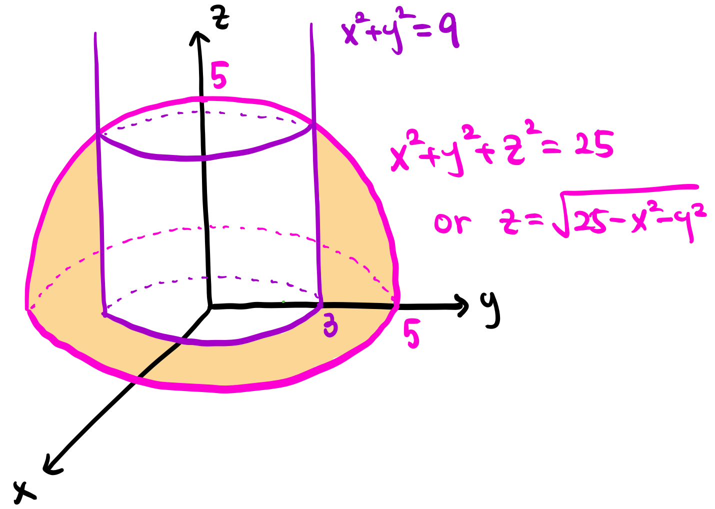
$(u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta)$

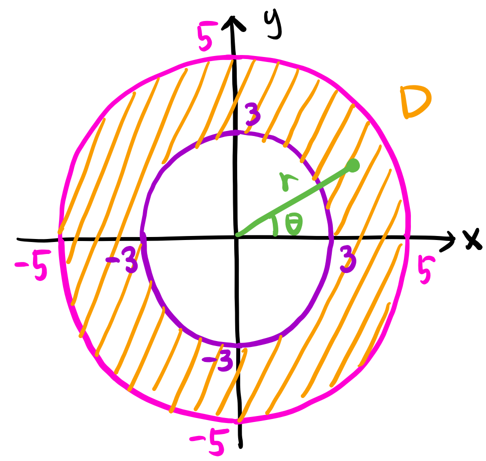
$$= \int_1^{-1} \frac{32}{5} u^2 \cdot (-1)\, du$$

$$= -\frac{32}{15} u^3 \Big|_{u=1}^{u=-1} = \boxed{\frac{64}{15}}$$

Ex Find the volume of the solid above the xy-plane bounded by the surfaces $x^2+y^2+z^2=25$ and $x^2+y^2=9$.

Sol





In polar coordinates, the domain D is given by

$0 \le \theta \le 2\pi$ and $3 \le r \le 5$.

The solid is under the graph $z=\sqrt{25-x^2-y^2}$ and above D.

$$\text{Volume} = \iint_D \sqrt{25-x^2-y^2}\, dA$$

Jacobian

$$= \int_0^{2\pi}\int_3^5 \sqrt{25-r^2}\cdot r\, dr\, d\theta$$

$x^2+y^2=r^2$

$(u = 25-r^2 \Rightarrow du = -2r\,dr)$

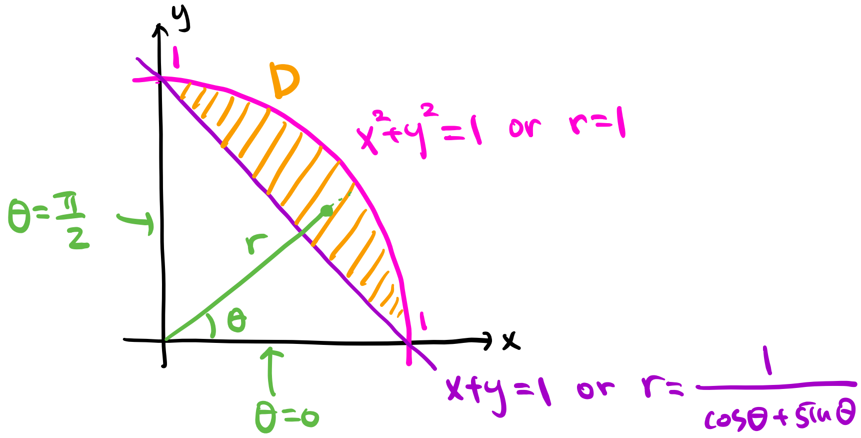
$$= \int_0^{2\pi}\int_{16}^{0} u^{1/2}\cdot\left(-\tfrac{1}{2}\right) du\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=16}^{u=0} d\theta$$

$$= \int_0^{2\pi} \frac{64}{3}\, d\theta = \boxed{\frac{128\pi}{3}}$$

Ex Evaluate $\iint_D \frac{x+y}{x^2+y^2}\, dA$ where D is the region given by $x^2+y^2 \le 1$ and $x+y \ge 1$.

Sol



In polar coordinates:

$x^2+y^2=1 \rightsquigarrow r^2=1 \rightsquigarrow r=1$

$x+y=1 \rightsquigarrow r\cos\theta + r\sin\theta = 1 \rightsquigarrow r = \frac{1}{\cos\theta+\sin\theta}$.

$\Rightarrow$ D is given by $0 \le \theta \le \frac{\pi}{2}$, $\frac{1}{\cos\theta+\sin\theta} \le r \le 1$.

$$\iint_D \frac{x+y}{x^2+y^2}\, dA = \int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \frac{r\cos\theta+r\sin\theta}{r^2}\cdot \overset{\text{Jacobian}}{r}\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \cos\theta+\sin\theta \, dr\, d\theta$$

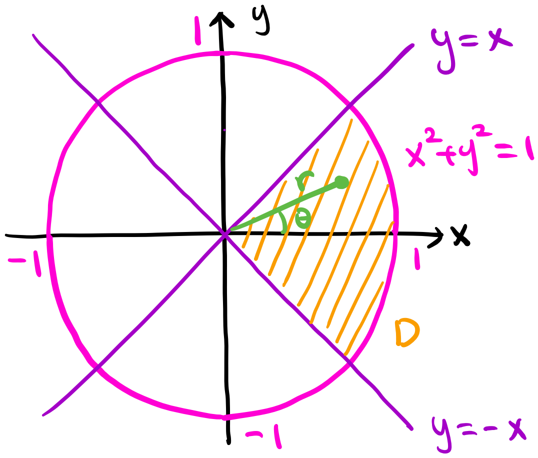
$$= \int_0^{\pi/2} \left(1-\frac{1}{\cos\theta+\sin\theta}\right)(\cos\theta+\sin\theta)\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta+\sin\theta-1\, d\theta$$

$$= (\sin\theta-\cos\theta-\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \boxed{2-\frac{\pi}{2}}$$

$\underline{\text{Ex}}$ Evaluate $\iint_D x^3y^2\,dA$ where D is the region given by $x^2+y^2 \le 1$ and $-x \le y \le x$.

$\underline{\text{Sol}}$



In polar coordinates:

$y=x \rightsquigarrow \frac{y}{x}=1 \rightsquigarrow \tan\theta=1 \rightsquigarrow \theta=\frac{\pi}{4}$ or $\cancel{\frac{5\pi}{4}}$

$y=-x \rightsquigarrow \frac{y}{x}=-1 \rightsquigarrow \tan\theta=-1 \rightsquigarrow \theta=-\frac{\pi}{4}$ or $\cancel{\frac{3\pi}{4}}$

$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$\Rightarrow D$ is given by $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and $0 \le r \le 1$.

Jacobian

$$\iint_D x^3y^2\,dA = \int_{-\pi/4}^{\pi/4}\int_0^1 r^3\cos^3\theta \cdot r^2\sin^2\theta \cdot r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\int_0^1 r^6\cos^3\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{r^7}{7}\cos^3\theta\sin^2\theta\Big|_{r=0}^{r=1}\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{7}\cos^3\theta\sin^2\theta\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{7}\cos\theta\cdot\cos^2\theta\sin^2\theta\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{7} \cos\theta \, (1-\sin^2\theta) \sin^2\theta \, d\theta$$

$(u = \sin\theta \Rightarrow du = \cos\theta \, d\theta)$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1}{7} (1-u^2) u^2 \, du = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{u^2}{7} - \frac{u^4}{7} \, du$$

$$= \left(\frac{u^3}{21} - \frac{u^5}{35} \right) \Bigg|_{u=-1/\sqrt{2}}^{u=1/\sqrt{2}} = \boxed{\frac{\sqrt{2}}{60}}$$